

CONSTANT-SPEED RAMPS

OSCAR M. PERDOMO

ABSTRACT. It is easy to show that if the kinetic coefficient of friction between a block and a ramp is μ_k and this ramp is a straight line with slope $-\mu_k$ then, this block will move along the ramp with constant speed. A natural question to ask is the following: Besides straight lines, are there more shapes of ramps such that a block will go down on the ramp with constant speed. Here we classify all the possible shape of these ramps and, **surprisingly** we will show that the planar ramps can be parametrized in term of elementary functions: trigonometric function, exponential functions and their inverses. They provide basic examples of explicitly parametrized arc-length parameter curves. A video explaining the main results in this paper can be found in one of the following URLs:

<http://www.youtube.com/watch?v=iBrvbb0efVk>

http://www.ccsu.edu/cf_media/index.cfm?g=871

1. INTRODUCTION

A basic computation shows that given a kinetic coefficient of friction μ_k , the angle of a linear ramp can be adjusted so that a block on this ramp will move down with constant speed. In order to cancel the gravity force, the friction force and the normal force, it is enough to take a ramp given by a line with slope $-\mu_k$, this is, we must take the inclination of the ramp to be θ_0 with $\tan(\theta_0) = \mu_k$. See Figure 1.1

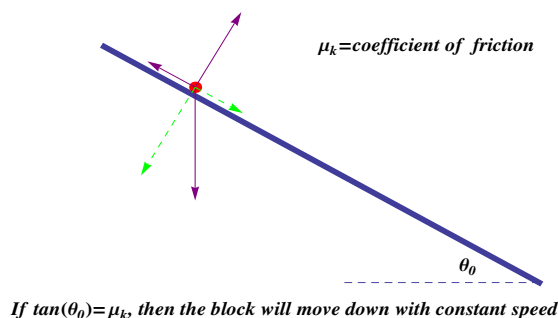


FIGURE 1.1. A block sliding down under the effect of gravity and the friction force. This figure shows the three forces acting on the block. The dashed forces are just a decomposition of the gravity force.

Let us assume that we want a block to move down with speed v_0 and we know that the kinetic coefficient of friction is μ_k . In this note we classify the shapes of all possible ramps on which this block will move down with speed v_0 . Here is how you would build this *constant-speed ramp*. Given v_0 and $\mu_k = \tan(\delta)$, let us define $a = \frac{g}{v_0^2 \sin \delta}$, ($g = 9.81 \frac{m}{s^2}$) and

$$(1.1) \quad \alpha(s) = \left(s + \frac{1}{a} \ln(1 + e^{-2as}), \frac{2}{a} \operatorname{arccot}(e^{-as}) \right)$$

This curve α has an U shape with two horizontal asymptotes, one at $y = 0$ and the second one at $y = \frac{\pi}{a}$. See Figure 1.2. We have:

If we clockwise rotate the curve α an angle δ (see Figure 1.3), then, the highest point divides this rotated curve in two ramps on which a block can move with constant speed v_0 under the assumption that the kinetic coefficient of friction is $\mu_k = \tan(\delta)$

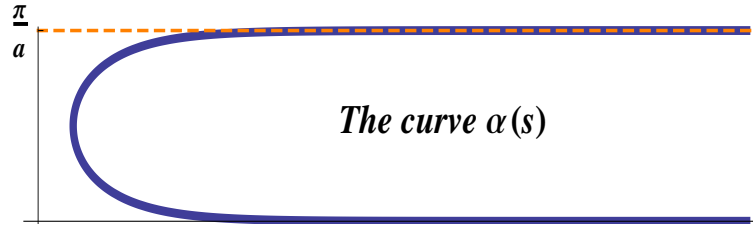


FIGURE 1.2. This is the graph of the curve α , it has two horizontal asymptotes that are separated $\frac{\pi}{a}$. The parameter s in the parametrization provided in the definition of α is arc-length.

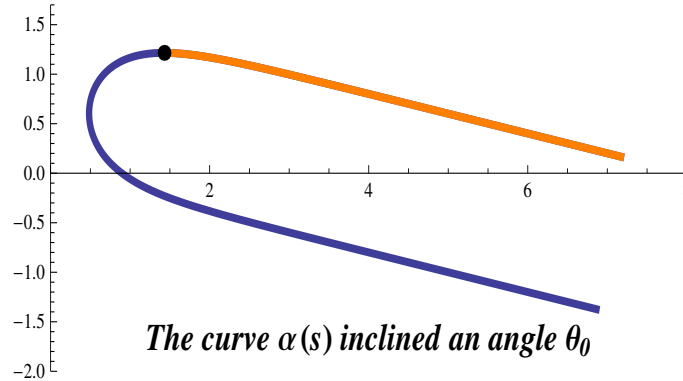


FIGURE 1.3. Graph of the two *constant-speed ramps*. At the highest point, the block with speed v_0 must be placed on the top of the ramp to the right and it must be placed upside down on the ramp to the left.

The sequence of pictures at the end of the paper shows how the motion on the ramps takes place when the desire speed is $5 \frac{m}{s}$ and $\mu_k = 0.5$. The picture also displays the three forces acting on the block under the assumption that the mass is 1 Kg. On the highest part of ramp the normal is zero, the normal in the upper part of the ramp is pointing down, the block does not fall down due to its speed.

In section 2 we provide a first proof for the classification of two dimensional ramps. In section 3 we provide a formal definition of ramp in order to state the result as a mathematical theorem. This section may be omitted. Section 4 deals with 3-dimensional ramps.

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2. A FIRST APPROACH

Let us find all the possible curves with the property that a block, sliding down on it, will move with constant speed $v > 0$ under the assumption that the kinetic coefficient of friction is $\mu = \tan(\delta)$ for some constant δ between 0 and $\frac{\pi}{4}$ radians. Let us start by assuming that this curve is parametrized by arc-length, this is, if $\alpha(s) = (x(s), y(s))$ denotes such a curve, then $|\alpha'(s)| = 1$. Under this assumption we can assume that for a smooth function $\theta(s)$ we have

$$x'(s) = \cos(\theta(s)) \quad \text{and} \quad y'(s) = \sin(\theta(s))$$

The function $\theta(s)$ will help us to describe the curve α . It is clear that the vector $n(s) = (-\sin(\theta(s)), \cos(\theta(s)))$ is a unit vector perpendicular to $\alpha'(s)$ and the chain rule give us that $\alpha''(s) = \theta'(s)n(s)$. Figure 2.1 shows these two vectors

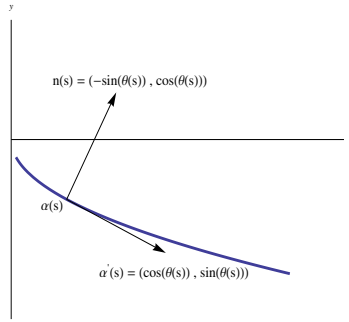


FIGURE 2.1. We are assuming that the parameter s is arc-length parameter. The vectors $\alpha'(s)$ and $n(s)$ are perpendicular.

Let us assume that $\beta(t) = \alpha(vt)$ describes the motion of the block. Since $s = vt$ and s is arc-length parameter, then the speed of the block is constant, it is v . Since we have that $\beta''(t) = v^2\alpha''(vt)$ then, Figure 2.2 shows the free body diagram for the problem of making the block move along α with constant speed v .

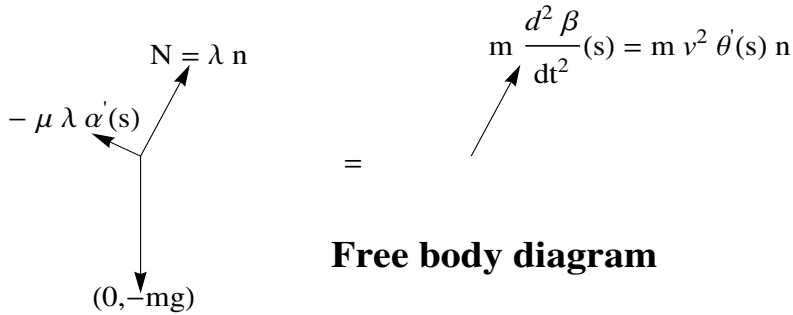


FIGURE 2.2. By Newton's second law, the sum of the normal force N plus the weight plus the friction force must be $m\beta''$.

As the free body shows the following equation must hold true.

$$(2.1) \quad mv^2\theta'(s)n(s) = \lambda(s)n(s) - \tan(\delta)\lambda(s)\alpha'(s) - (0, mg)$$

By doing the inner product of both sides of the equation 2.1 with the vector $\alpha'(s)$ we obtain that

$$\lambda(s) = -mg \cot(\delta) \sin(\theta(s))$$

Likewise, by doing the inner product of both sides of the equation 2.1 with the vector $n(s)$, we obtain that $mv^2\theta' = \lambda - mg \cos(\theta)$ and therefore, if $a = \frac{g}{v^2 \sin(\delta)}$, then

$$(2.2) \quad \theta'(s) = \frac{-g}{v^2 \sin(\delta)} \sin(\theta(s) + \delta) = -a \sin(\theta(s) + \delta)$$

Since the differential equation (2.2) does not have the variable s , then, all the solutions differ by a horizontal translation. This is, if $\theta(s)$ is a solution, then $\theta(s + c)$ is also a solution for every real number c . Due to the geometry of our problem we do not need to consider an integrating constant, since just one solution will give us all the solutions $\alpha(s)$. Recall that the equilibrium solution of the differential equation (2.2) is $\theta(s) = -\delta$ for all s . This solution corresponds to the case $\alpha(s) = (\cos(\delta)s, -\sin(\delta)s)$ which is the straight line ramp shown in Figure 1.1. When we solve this differential equation by separation of variables we notice that we need to integrate the function $\csc(\theta + \delta)$. Instead of using the classical formula $\int \csc(u) du = -\ln(\csc(u) + \tan(u))$ we will use the formula $\int \csc(u) du = \ln(\tan(\frac{u}{2}))$ which led us to the formula

$$\theta(s) = -\delta + 2 \arctan(e^{-as})$$

It is clear that if $\gamma(s) = (z(s), w(s))$ denotes a counterclockwise rotation of δ radians of the curve $\alpha(s)$, then

$$z'(s) = \cos(2 \arctan(e^{-as})) \quad \text{and} \quad w'(s) = \sin(2 \arctan(e^{-as}))$$

Integrating the equations above we obtain that

$$z(s) = s + \frac{\ln(1 + e^{-2as})}{a} \quad \text{and} \quad w(s) = \frac{2}{a} \operatorname{arccot}(e^{-as})$$

The curve $(z(s), w(s))$ is shown in Figure 1.2. As mentioned in the introduction, the solution that we are looking for is a clockwise rotation of the curve γ by an angle δ . In the next section we give a more detailed explanation of this solution.

3. UNDERSTANDING THE SOLUTION OF THE ODE. A MATHEMATICAL DEFINITION OF RAMP

Let us start this section with a definition.

Definition 1. We will say that a curve $\gamma : [a_1, a_2] \rightarrow \mathbf{R}^2$ is regular if $\gamma'(t)$ never vanishes. We say $n : [a_1, a_2] \rightarrow \mathbf{R}^2$ is a normal of the curve γ if $n(t)$ has length 1 and the inner product of $n(t)$ and $\gamma'(t)$ is zero, this is, $n(t) \cdot \gamma'(t) = 0$ for all t .

Definition 2. A ramp in the plane \mathbf{R}^2 is an ordered pair (γ, n) where $\gamma : [a_1, a_2] \rightarrow \mathbf{R}^2$ is a regular curve and $n : [a_1, a_2] \rightarrow \mathbf{R}^2$ is a normal. We will interpret the ramp (γ, n) as a portion of the plane whose boundary contains γ and its outer normal vector along is n .

Example 1. (γ_1, n_1) where $\gamma_1, n_1 : [0, \pi] \rightarrow \mathbf{R}^2$ are given by $\gamma_1(t) = 5(\cos(t), \sin(t))$ and $n_1(t) = (\cos(t), \sin(t))$ is an example of a ramp. This ramp is shown in the first two images in Figure 3.1. (γ_2, n_2) where $\gamma_2, n_2 : [0, \pi] \rightarrow \mathbf{R}^2$ are given by $\gamma_2(t) = 5(\cos(t), \sin(t))$ and $n_2(t) = -(\cos(t), \sin(t))$ is an example of a ramp. This ramp is shown in the last two images in Figure 3.1.

The following definition is based on Newton's second law.

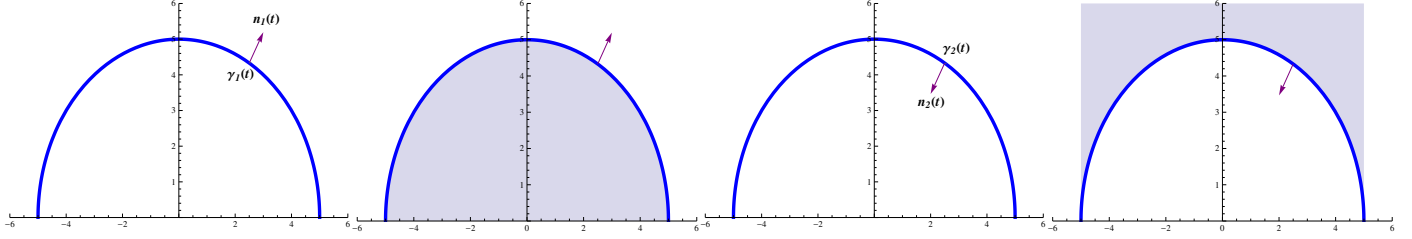


FIGURE 3.1. A normal vector of a curve helps us to define the part of a curve where we want a object to slide on a ramp

Definition 3. A ramp with external force is a triple (γ, n, F) where $F : [a_1, a_2] \rightarrow \mathbf{R}^2$ is a smooth function and (γ, n) is a ramp. Given a positive number m , a **solution of the ramp with external force** (γ, n, F) is given by a curve $\beta(t) = \gamma(h(t))$ (Notice that β is just a re-parametrization of the curve γ and it is completely determined by the function $h : [a_1, a_2] \rightarrow \mathbf{R}$) and a nonnegative function $\lambda : [a_1, a_2] \rightarrow \mathbf{R}$ such that

$$F(h(t)) + \lambda(t)n(h(t)) = m\beta''(t)$$

Remark 3.1. In Definition 3, the term $\lambda(t)n(t)$ represents the force that the surface of the ramp is doing on the object that moves on the ramp under the action of the external force F . By Newton's third law, $-\lambda(t)n(t)$ is the force that the object is doing to the ramp. The condition $\lambda \geq 0$ is needed so that the object stays on the ramp.

Example 2. If (γ, n) is a ramp, $m > 0$ and $F(t) = (0, -9.81m)$, then the triple (γ, n, F) represents the action of the gravity acting on a particle with mass m that moves on the ramp without friction.

Example 3. Let us consider the ramp (γ, n) where $\gamma, n_2 : [-1.5, 1.5] \rightarrow \mathbf{R}^2$ are given by $\gamma(t) = (t, t^2)$ and $n(t) = (\frac{2t}{\sqrt{4t^2+1}}, \frac{-1}{\sqrt{4t^2+1}})$. If $F(t) = (0, -9.81)$ then, the ramp with external force (γ, n, F) has no solution. Figure 3.2 shows this ramp.

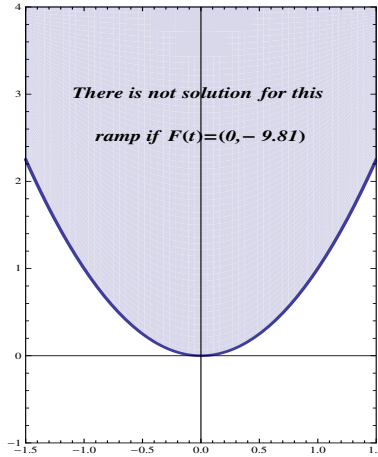


FIGURE 3.2. Some ramps with external force do not have a solution.

Definition 3 provides the definition of the solution of a particle moving on the ramp under the action of the force F . In the case that F denotes all the forces acting on the particle that moves on the ramp *but* the friction force, the definition of solution needs to be changed to:

Definition 4. Given a ramp with external force (γ, n, F) defined on the interval $[a_1, a_2]$ and two positive numbers $\mu < 1$ and m , a **solution of the ramp with external force (γ, n, F) and kinetic coefficient of friction μ** is given by a curve $\beta(t) = \gamma(h(t))$ and a nonnegative function $\lambda : [a_1, a_2] \rightarrow \mathbf{R}$ such that

$$F(h(t)) + \lambda(t)n(h(t)) - \mu\lambda(t)\frac{\beta'(t)}{|\beta'(t)|} = m\beta''(t)$$

Remark 3.2. Let us define an *elementary function* to be a function of one variable with real values built from a finite number of trigonometric, exponential, constant, n^{th} functions and their inverses, through composition and using the four basic elementary operation $(+, -, \times, \div)$. When we want to define basic examples of curves defined by arc-length parameter, that is, if we want to define $\gamma(s) = (x(s), y(s))$ such that $x'(s)^2 + y'(s)^2 = 1$ for all t , then, the first thing that comes to our mind is to find easy possibilities for $x'(t)$ and $y'(t)$. The easiest one would be a pair of numbers c_1 and c_2 such that $c_1^2 + c_2^2 = 1$. If we make $x'(s) = c_1$ and $y'(s) = c_2$ then after an easy integration we obtain that the curve γ is a straight line. If we want to use the fact that $\cos(as)^2 + \sin(as)^2 = 1$ and we decide to make $x'(s) = \cos(as)$ and $y'(s) = \sin(as)$, then, after an easy integration we obtain that γ should be a circle. The most important curves in this paper are those that are obtained by using the identity

$$\tanh(t)^2 + \operatorname{sech}(t)^2 = 1 \quad \text{for all } t$$

That is, we will be using curves that satisfy $x'(s) = \tanh(as)$ and $y'(s) = \operatorname{sech}(as)$.

The following lemma is a direct computation and provides a definition of the curve α whose graph is shown in Figure 1.2

Lemma 3.3. For any non zero real number a , the curve

$$(3.1) \quad \alpha(s) = (x(s), y(s)) = \left(s + \frac{1}{a} \ln(1 + e^{-2as}), \frac{2}{a} \operatorname{arccot}(e^{-as}) \right)$$

is an arc-length parametrized curve. Moreover,

$$x'(s) = \tanh(as) \quad \text{and} \quad y'(s) = \operatorname{sech}(as)$$

Lemma 3.4. For any positive number δ smaller than $\frac{\pi}{2}$, if α_δ represents the curve obtained by rotating an angle of δ radians the curve $\alpha = (x(s), y(s))$ defined in Lemma 3.3, this is, if

$$(3.2) \quad \alpha_\delta(s) = (x_\delta(s), y_\delta(s)) = (\cos(\delta)x(s) + \sin(\delta)y(s), -\sin(\delta)x(s) + \cos(\delta)y(s))$$

then, the maximum value for the second entry of α_δ is achieved when $s = \frac{\operatorname{arcsinh}(\cot(\delta))}{a}$. Also, we have that

$$(3.3) \quad \alpha_\delta''(s) \cdot (y_\delta'(s), -x_\delta'(s)) = a \operatorname{sech}(as)$$

The graph of the curve α_δ is shown in Figure 1.3.

Proof. A direct computation using Lemma 3.3 shows that $y'_\delta(s) = -\sin(\delta)\tanh(as) + \cos(\delta)\operatorname{sech}(as)$, then we can see that the only solution of the equation $y'_\delta(s) = 0$ is $s = \frac{\operatorname{arcsinh}(\cot(\delta))}{a}$. In order to prove the identity 3.3 we point out that since the inner product is invariant under rotations, this identity is equivalent to show that $\alpha''(s) \cdot (y'(s), -x'(s)) = a \operatorname{sech}(as)$ which follows because,

$$x''(s)y'(s) - y''(s)x'(s) = a \operatorname{sech}^3(as) + a \tanh^2(as) \operatorname{sech}(as) = a \operatorname{sech}(as) (\operatorname{sech}^2(as) + \tanh^2(as)) = a \operatorname{sech}(as)$$

□

Remark 3.5. In the previous proof we got that $y'_\delta(s) = -\sin(\delta)\operatorname{sech}(as)(\sinh(as) - \cot(\delta))$. It follows that $y'_\delta(s) > 0$ if $s < s_0$ and $y'_\delta(s) < 0$ if $s > s_0$.

Definition 5. For any positive number δ smaller than $\frac{\pi}{2}$ and any $a > 0$ we define the ramps $(\gamma_\delta, n_\delta)$ and $(\tilde{\gamma}_\delta, \tilde{n}_\delta)$ by

$$\gamma_\delta(s) = \alpha_\delta(s_0 - s) \quad \text{and} \quad n_\delta(s) = (y'_\delta(s_0 - s), -x'_\delta(s_0 - s))$$

and

$$\tilde{\gamma}_\delta(s) = \alpha_\delta(s + s_0) \quad \text{and} \quad \tilde{n}_\delta(s) = (-y'_\delta(s + s_0), x'_\delta(s + s_0))$$

where $s_0 = \frac{\operatorname{arcsinh}(\cot(\delta))}{a}$ and the maps γ_δ and n_δ are defined in Lemma 3.4 and all the function $\alpha_\delta, n_\delta, \tilde{\alpha}_\delta, \tilde{n}_\delta$ are defined on the interval $[0, \infty)$. These two ramps are shown in Figure 3.3.

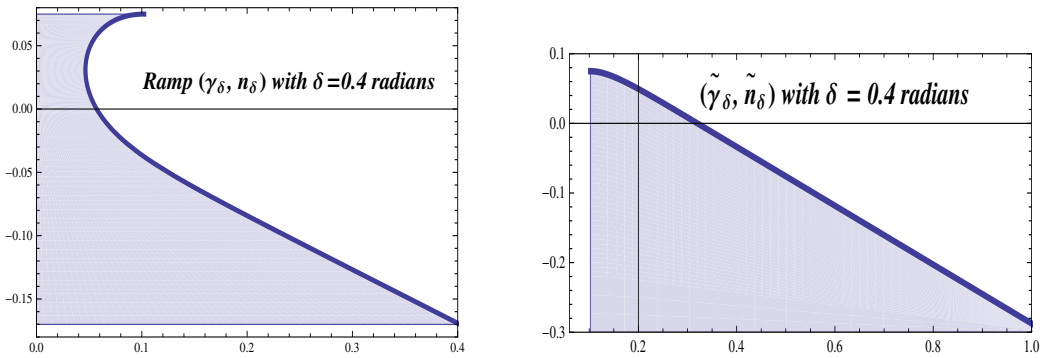


FIGURE 3.3. The two ramps define in Definition 5

Let us state and prove the main theorem in this section.

Theorem 3.6. Given an angle δ between 0 and $\frac{\pi}{4}$ radians and a positive speed $v > 0$. Let us consider $\mu = \tan(\delta)$, $a = \frac{g}{v^2 \sin \delta}$ and $F(t) = (0, -mg)$ with $g = 9.81$. We have

a.) For the ramp $(\gamma_\delta, n_\delta)$ given in Definition 5,

$$\beta(t) = \gamma_\delta(vt) \quad \text{and} \quad \lambda(t) = mg \cot(\delta) y'_\delta(s_0 - vt)$$

is a solution for the ramp with external force $(\gamma_\delta, n_\delta, F)$ and kinetic coefficient of friction $\tan(\delta)$.

b.) For the ramp $(\tilde{\gamma}_\delta, \tilde{n}_\delta)$ given in Definition 5,

$$\beta(t) = \tilde{\gamma}_\delta(vt) \quad \text{and} \quad \lambda(t) = -mg \cot(\delta) y'_\delta(s_0 + vt)$$

is a solution for the ramp with external force $(\tilde{\gamma}_\delta, \tilde{n}_\delta, F)$ and kinetic coefficient of friction $\tan(\delta)$.

c.) Since in both cases $|\beta'(t)| = v$, then these motions have constant speed.

Proof. Let us prove part **a.)**. First of all, notice that as it is required by Definition 4, $\lambda > 0$ by remark 3.5. From the definition of β we obtain that

$$\beta'(t) = -v\alpha'_\delta(s_0 - vt) \quad \text{and} \quad \beta''(t) = v^2 \alpha''_\delta(s_0 - vt)$$

therefore the equation

$$F(h(t)) + \lambda(t)n(h(t)) - \mu\lambda(t)\frac{\beta'(t)}{|\beta'(t)|} = m\beta''(t)$$

is equivalent to

$$(3.4) \quad (0, -mg) + \lambda(t)n_\delta(vt) + \tan(\delta)\lambda(t)\alpha'_\delta(s_0 - vt) = mv^2\alpha''_\delta(t)$$

Notice that the vector $\mathbf{u}_1 = n_\delta(vt)$ and $\mathbf{u}_2 = \alpha'_\delta(s_0 - tv)$ form an orthonormal basis. Therefore in order to prove equation (3.4) it is enough to prove that the dot product with \mathbf{u}_1 and \mathbf{u}_2 of the left hand side (LHS) and right hand side (RHS) of the equation is the same. The dot product of the LHS of equation (3.4) with \mathbf{u}_1 is equal to

$$\begin{aligned} mg x'_\delta(s_0 - vt) + \lambda(t) &= mg x'_\delta(s_0 - vt) + \cot(\delta)y'_\delta(s_0 - vt) \\ &= mg (\cos(\delta)\tanh(a(s_0 - vt)) + \sin(\delta)\operatorname{sech}(a(s_0 - vt))) + \\ &\quad mg \cot(\delta) (-\sin(\delta)\tanh(a(s_0 - vt)) + \cos(\delta)\operatorname{sech}(a(s_0 - vt))) \\ &= \frac{mg}{\sin(\delta)} \operatorname{sech}(a(s_0 - vt)) \end{aligned}$$

Using Equation (3.3) we get that the dot product of the RHS of equation (3.4) with \mathbf{u}_1 is equal to

$$mv^2 a \operatorname{sech}(a(s_0 - vt)) = \frac{mg}{\sin(\delta)} \operatorname{sech}(a(s_0 - vt))$$

Therefore $\text{LHS} \cdot \mathbf{u}_1 = \text{RHS} \cdot \mathbf{u}_1$. It is easy to check that the dot product of the RHS of equation (3.4) with \mathbf{u}_2 vanishes. On the other hand, the dot product of the LHS of the equation (3.4) with \mathbf{u}_2 equals to

$$-mg y'_\delta(s_0 - vt) + v \tan(\delta) \lambda = -mg y'_\delta(s_0 - vt) + \tan(\delta) mg \cot(\delta) y'_\delta(s_0 - vt) = 0$$

Therefore part **a.)** follows. Part **b.)** is similar. Part **c.)** follows because $|\alpha'| = 1$ and since α_δ is a rotation of α then $|\alpha'_\delta| = 1$, then, $\beta'(t) = -v\alpha'_\delta(s_0 - vt)$ and $|\beta'(t)| = v$

□

4. 3-DIMENSIONAL RAMPS

In this section we describe ramps in the space on which an object can move with constant speed v_0 under the assumption that the kinetic coefficient of friction is μ . We will prove that fixing v_0 and μ , there are as many different ramp as continuous unit tangent vector fields in the south hemisphere. In the correspondence that we will establish, the ramp that we defined in section 2 corresponds to a particular choice of the a tangent vector field.

For curves that represent planar ramps, we got a description of them by studying first their velocity vector. Recall that the function $\theta(s)$ was used to described a point in the lower part of the semicircle that represented a possible velocity vector of the curve that describe the ramp. See Figure 4.1. When this curve is free to move in the whole 3-dimensional space, the tangent unit vector to the curve lies in the unit sphere and since the block must go down the ramp we can assume that the tangent unit vector of the ramp lies in the shouth hemisphere. See Figure 4.1.

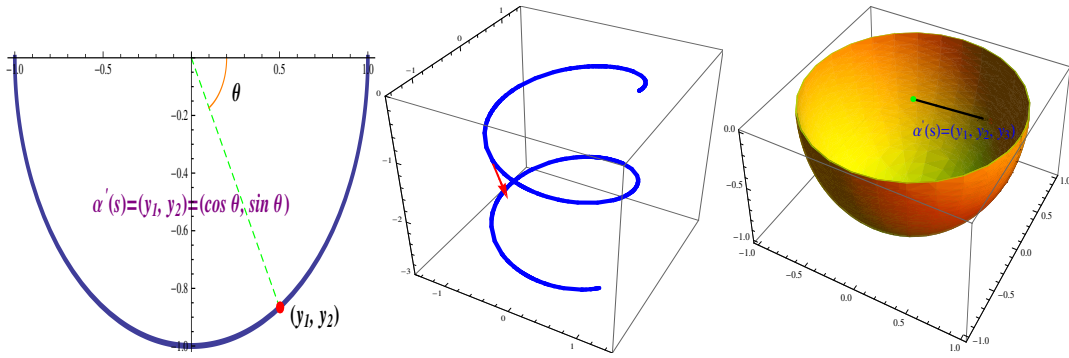


FIGURE 4.1. The unit tangent vector of a possible ramp in the plane lies in the lower part of the unit semicircle, if the ramp is free to move in the whole space, then the unit tangent vector lies in the lower hemisphere of the unit sphere

in order to completely determine the ramp we need to specify the desire direction of the normal to the ramp, see Figure 4.2. We can do this by choosing a unit normal vector field $H(y)$ defined in the lower hemisphere and requesting the normal vector of the ramp to be equal to $N(y)$ anytime the tangent unit vector to the ramp is the vector y . See Figure 4.3.

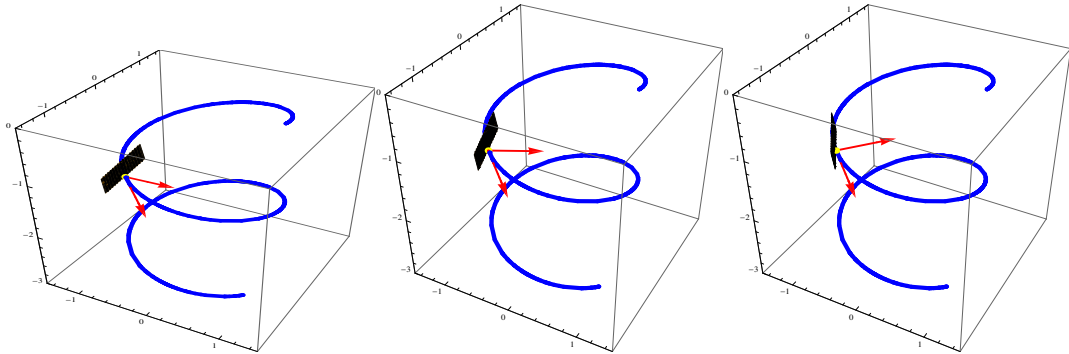


FIGURE 4.2. To built a ramp we need to decide about the normal direction at every point in the ramp.

We will prove that for any continuous choice of a normal field $N(s)$ in the lower hemisphere, there exist a family of ramp on which a motion with constant speed is possible under the effect of the gravity force and the force of friction. More precisely we have.

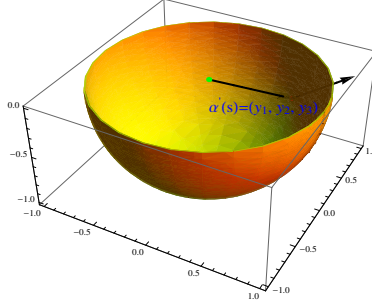


FIGURE 4.3. We will make the choice of the normal direction on the ramp to depend on the unit tangent vector.

Theorem 4.1. *Let $\Sigma = \{(y_1, y_2, y_3) \in \mathbf{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1 \text{ and } y_3 \leq 0\}$ denote the south hemisphere and let $N : \Sigma \rightarrow \mathbf{R}^3$ be any unit tangent vector field. This is, for any $y \in \Sigma$, $N(y)$ has norm 1 and $N(y)$ is perpendicular to y . For any positive numbers $v, \mu < 1$ and any $y_0 \in \Sigma$ there exists a unique curve $\alpha(s)$ parametrize by arc-length, such that $\alpha(0) = (0, 0, 0)$, $\alpha'(0) = y_0$ and the motion along $\beta(t) = \alpha(vt)$ in the ramp given by*

$$R(s, r) = \alpha(s) + r \alpha'(s) \times N(\alpha(s))$$

represents a motion that is a solution of Newton's second law under the assumption that the only forces action on this moving particle are gravity and a friction force with kinetic constant coefficient μ . Recall that since the parameter s is arc-length parameter, then $|\beta'(t)| = v$ for all t ; this is, the motion has constant speed v .

Proof. Recall that we are denoting by $u \cdot w$ the dot (or inner) product of the two vectors u and w . A direct computation shows that if $e_3 = (0, 0, 1)$, then $e_3^T : \Sigma \rightarrow \mathbf{R}^3$ given by $e_3^T(y) = e_3 - (e_3 \cdot y)y$ is a tangent vector field (notice that $e_3^T(y)$ must be in the tangent space $T_y \Sigma$ because $e_3^T(y) \cdot y = 0$)

Let $\lambda : \Sigma \rightarrow \mathbf{R}$ be the function given by

$$\lambda(y) = -\frac{g}{\mu} e_3 \cdot y = -\frac{g}{\mu} y_3$$

Notice that since $y_3 \leq 0$ then $\lambda \geq 0$ and λ only vanishes on the boundary of Σ , (recall that the boundary of Σ is given by the equation $y_3 = 0$). Let us consider the tangent vector field

$$(4.1) \quad X = -\frac{1}{v^2} (ge_3^T - \lambda N(y)) = -\frac{g}{v^2} \left(e_3^T + \frac{y_3}{\mu} N(y) \right)$$

Notice that along points in the boundary of Σ , $X(y) = -\frac{g}{v^2} e_3$ points toward Σ . It follows that any integral curve of the tangent vector field X remain in Σ . Let $\gamma(s)$ be the integral curve of the vector field X that satisfies $\gamma(0) = y_0$. We will prove the Theorem by showing the curve $\alpha(s)$ given by

$$\alpha(s) = \int_0^s \gamma(u) du$$

satisfies all the conditions. Clearly, $\alpha(0) = (0, 0, 0)$ and $\alpha'(0) = \gamma(0) = y_0$. We also have that s is arc-length parameter because $|\alpha'(s)| = |\gamma(s)| = 1$. A direct computation show that the normal vector of the surface (or ramp) R along the curve α (along points in the surface with $r = 0$) is given by $N(\alpha(s))$. We will now proof that Newton's second law holds true for $\beta(t) = \alpha(vt)$. Notice that

since v is constant, then $\beta'(t) = v\alpha'(vt) = v\gamma(s)$ and $\beta''(vt) = v^2\gamma'(vt) = v^2\gamma'(s)$. If m denotes the mass of the particle and λ is defined as in the beginning of the proof, we have that

$$\begin{aligned} m\beta''(t) = mv^2\gamma'(s) &= -mv^2\frac{1}{v^2}(ge_3^T(\gamma) - \lambda N(\gamma)) \\ &= m(-ge_3 + g(e_3 \cdot \gamma)\gamma - \frac{g}{\mu}(e_3 \cdot \gamma)N(\gamma(t))) \\ &= -gme_3 + m\lambda N(\gamma(s)) - m\lambda\mu\gamma(s) \end{aligned}$$

From the equation above we conclude that then normal force made from the ramp to the block has magnitude $m\lambda$. Notice that the last equation above is Newton's second law.

It is clear the the form of the ramp depends on the desire constant speed that we want on the ramp. For example if the block is to travel outside down on some portion of the ramp and the speed v is not much then the curvature of that portion of the ramp must be big so that the block does not fall off the ramp. The following corollary explains how to change the ramp if we want to change the constant speed or if we want to change the gravity force. \square

Corollary 4.2. *a.) If a ramp $R \subset \mathbf{R}^3$ does the work of allowing a block to move down with constant speed v and gravity g , then, for any positive κ the ramp $\kappa R = \{\kappa(x, y, z) : (x, y, z) \in R\}$ does the work of allowing a block to move down with constant speed $\sqrt{\kappa}v$ and gravity g .*

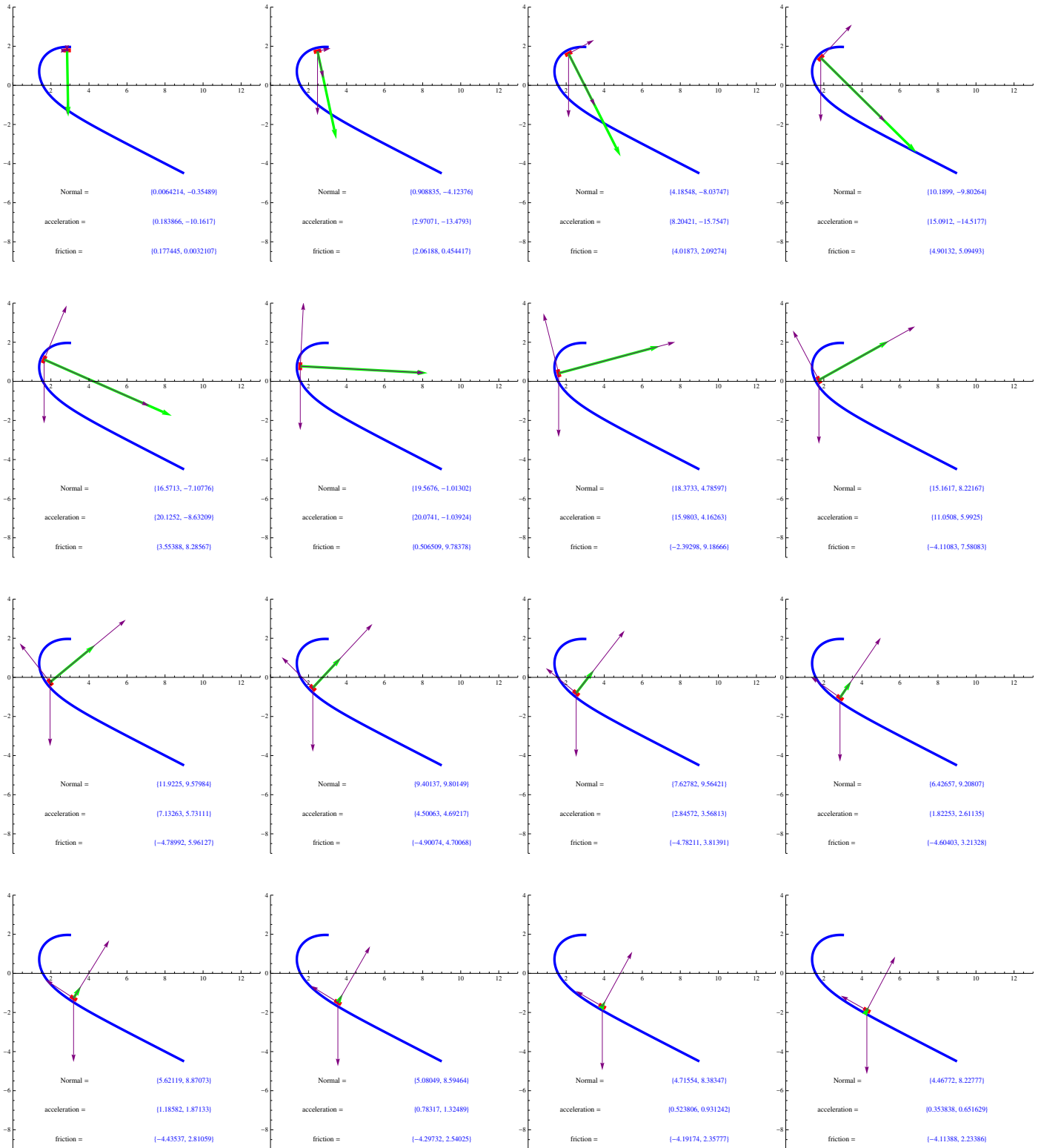
b.) If a ramp $R \subset \mathbf{R}^3$ does the work of allowing a block to move down with constant speed v and gravity g , then, for any positive κ the ramp $\kappa R = \{\kappa(x, y, z) : (x, y, z) \in R\}$ does the work of allowing a block to move down with constant speed v and gravity $\frac{g}{\kappa}$.

Proof. This corollary is a consequence of Equation (4.1). We can check that if $\gamma(t)$ is an integral curve of the vector field X , and $\alpha(s) = \int_0^s \gamma(u) du$, then $\tilde{\gamma}(\tau) = \gamma(\frac{\tau}{\kappa})$ is a solution of the vector field $\frac{1}{\kappa}X$, which can be interpreted as either the vector field coming from the tangent unit vector field $N(y)$ with velocity $\sqrt{\kappa}v$ and gravity g or it can be interpreted as the vector field coming from the tangent unit vector field $N(y)$, with velocity v and gravity $\frac{g}{\kappa}$. The result follows by noticing that

$$\tilde{\alpha}(s) = \int_0^s \tilde{\gamma}(u) du = \int_0^s \gamma(\frac{u}{\kappa}) du = \kappa \int_0^{\frac{s}{\kappa}} \gamma(v) dv = \kappa \alpha(\frac{s}{\kappa})$$

\square

Remark 4.3. If a ramp R on Earth has the property than an object will fall down with constant speed v , then, if we dilate this ramp by a factor of 6, the same block will move down with the same constant speed v on this dilated ramp when it is placed on the Moon.



Current address: Department of Mathematics, Central Connecticut State University, New Britain, CT 06050,

E-mail address: perdomoosm@ccsu.edu

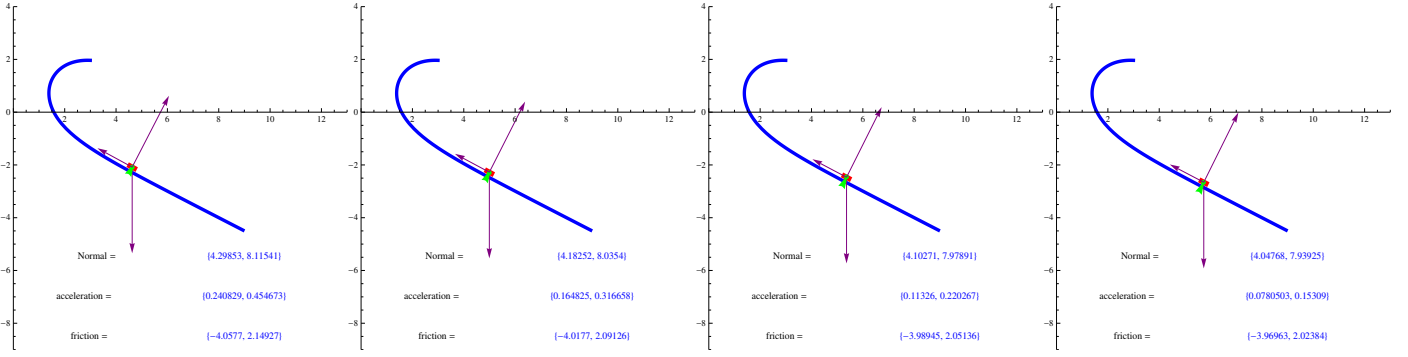


FIGURE 4.4. Motion on the longest parts of the ramps

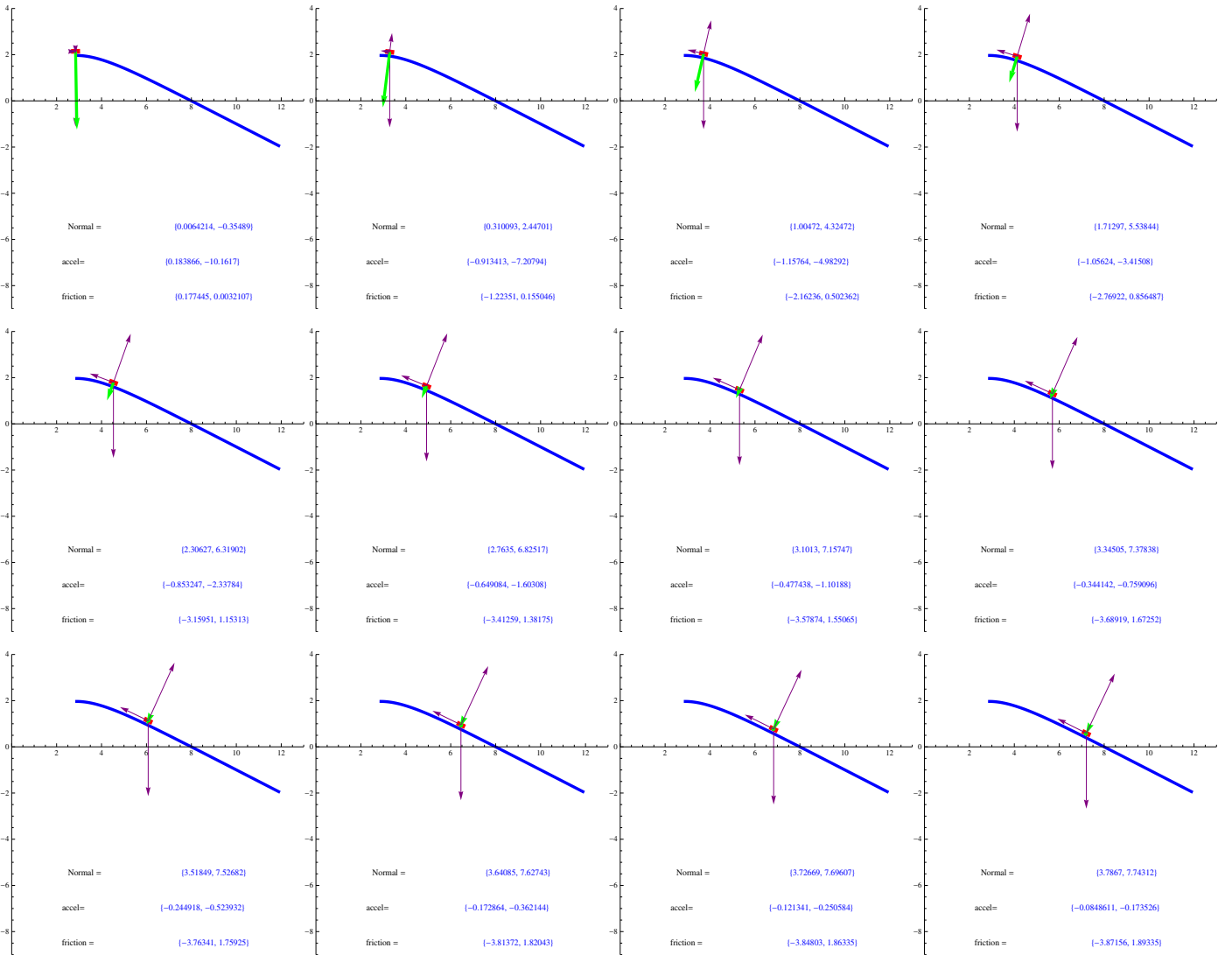


FIGURE 4.5. Motion on the shortest parts of the ramps